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Bernstein Operational Matrix for Solving Boundary Value Problems

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Abstract

This paper outlines a numerical method called the Bernstein operational matrix of derivative (BOMD) of order two and order three with the approach of the Chebyshev collocation technique to solve boundary value problems (BVP). BOMD with suitable collocation points is implemented to solve the BVP using the linear combination of Bernstein polynomials with unknown coefficients to approximate the solutions. The derivatives featured in the problem sets will be approximated by utilizing the matrix. The subsequent examination involves a mathematical analysis of the proposed method, including evaluating its order, absolute error metrics and comparative assessments with alternative methodologies. Four problems involving linear and non-linear equations and systems, along with practical real-world problems, are addressed to assess the reliability of the proposed method.

Keywords: operational matrix of derivatives; collocation method; boundary value problem; Bernstein polynomial basis.

1 Introduction

Boundary value problems (BVP) are the mathematical models in various real-world problems [12] extensively investigated in a wide range of studies. Some real-world problems include hydrodynamics [23], wave propagation [16] and in the theory of elastic stability [7]. A BVP refers to a set of differential equations (DE) for which solutions are designated at multiple points [12]. The solutions of BVP can be determined by using the exact solutions of the problems. However, not all BVP acquire the exact solutions. Therefore, the solutions can be computed using approximate solutions with various numerical methods.

BVP has been arising in various aspects of our lives to the extent that they are nearly unavoidable [12]. There are many studies on various numerical schemes, such as finite difference method [4] and forward time-centered space scheme to solve BVP. However, it is essential to point out that most research published to date concerning exact and numerical solutions of DE are devoted to the initial value problems or BVP for a specific case. Therefore, this research proposes an effective and simple operational method for the solution of equations and systems of linear and non-linear BVP.

In previous studies, researchers have employed different types of operational matrices and collocation methods to solve various BVP. Since 2013, the Bernoulli [24, 21, 30] and Legendre [25, 20] (matrix and collocation) have been used to solve non-linear DE and multi-Pantograph delay BVP. Other numerical methods such as Fully Jacobi-Galerkin [8] and Chebyshev Petrov-Galerkin [28] are also used to solve many types of time-fractional equations such as heat equation [29], diffusion equation [15], sub-diffusion equation [9] and KdV-Burgers' equation [27].

A method based on Bernstein operational matrix (BOM) of integration addresses linear time-varying systems described by DE and determines the inverse Laplace transform of specific functions [14]. Some of the problems include variable order fractional optimal control problems [10], linear and non-linear delay DE [2] and time-fractional order telegraph equations [1].

By reviewing the solution techniques employed in previous studies, this study intends to recognize the importance and challenges of constructing new numerical methods for solving BVP. Past studies in this field had focused on single equation of BVP problems. However, there is less attention on the systems of BVP. Some algorithms are complicated and use plenty of CPU time to compute. The gaps in previous studies have shown that the numerical methods for BVP is advancing in the field. Therefore, this study attempts to provide a more effective, simple and fast numerical algorithms for solving equations of BVP and systems of equations of BVP.

This paper applies the Bernstein operational matrix of derivative (BOMD) method to solve the BVP. Numerical results are compared with those obtained using other iterative methods from previous research as a test of the efficiency of the proposed method. The advantage of this method is by determining the approximate solutions of non-linear equations for which exact solutions cannot be obtained.

The following is the structure of this paper. Section 2 deals with the proposed method derivation, namely the BOMD method constructed by implementing Bernstein interpolation polynomial. The application of the method in solving both linear and non-linear equations and systems, along with a real-world problem is discussed in Section 3. Section 4 presents the concluding remarks and recommendations for future works.

2 Methodology

2.1 Bernstein polynomials

The n^{th} degree of Bernstein basis polynomials as mentioned in [11, 19] is defined on the interval $[0,1]$ as

$$B_i^n(x) = \begin{pmatrix} n \\ i \end{pmatrix} x^i(1-x)^{n-1}, \quad 0 \le i \le n. \tag{1}$$

Using a recursive definition [18], Bernstein polynomials over $[0,1]$ can be generated such that

$$B_i^n(x) = (1-x)B_i^{n-1}(x) + xB_{i-1}^{n-1}(x), \tag{2}$$

where $B_{-1}^{n-1}(x) = 0$ and $B_n^{n-1}(x) = 0$. Any polynomial function of degree n can be expressed by the linear combination of the basis functions

$$y(x) = \sum_{i=0}^{n} C_i B_i^n(x) = C^T B(x), \quad n \ge 1, \tag{3}$$

for equation, and for systems of equations it can be defined as

$$y_{j,n}(x) = c_{0,j}b_0^n(x) + c_{1,j}b_1^n(x) + \ldots + c_{n,j}b_n^n(x) = {C_j}^T B(x), \quad n \ge 1, \quad j = 1, 2. \tag{4}$$

C is the coefficients vector and $B(x)$ denote the Bernstein vector, where

$$C^T = [c_0, c_1, c_2, \ldots, c_n], \tag{5}$$

and

$$B(x) = [b_0^n(x), b_1^n(x), b_2^n(x), \ldots, b_n^n(x)]^T. \tag{6}$$

Due to its numerical stability, the Bernstein polynomials are helpful in the practical computations of numerical solutions [6]. Two fundamental properties of the Bernstein polynomials are the positivity and the partition of unity for all real x in the interval $[0,1]$, that is $\sum_{i=0}^{n} B_i^n(x) = 1$.

2.2 Bernstein operational matrix of derivatives

This section derives the explicit formula of the BOMD of the n^{th} degree. Suppose that D is an $(n\times 1)+(n\times 1)$ order of operational matrix of derivative [26], then the $B(x)$ derivatives is written as

$$\frac{d}{dx}B(x) = D^{(1)}B(x). \tag{7}$$

From [26], $D^{(1)}$ is specified as $D^{(1)} = AH\hat{B}$ and

$$\begin{aligned} A_{i+1} = \Bigg[& \overbrace{0, 0, \ldots, 0}^{\text{repeated } i \text{ times}}, (-1)^0 \begin{pmatrix} n \\ i \end{pmatrix} \begin{pmatrix} n-i \\ 0 \end{pmatrix}, (-1)^1 \begin{pmatrix} n \\ i \end{pmatrix} \begin{pmatrix} n-i \\ 1 \end{pmatrix}, \\ & \ldots, (-1)^{n-i} \begin{pmatrix} n \\ i \end{pmatrix} \begin{pmatrix} n-i \\ n-i \end{pmatrix} \Bigg], \end{aligned} \tag{8}$$

$$H = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & n \end{bmatrix}_{(n+1)\times(n)}, \quad \hat{B} = \begin{bmatrix} A^{-1}_{[1]} \\ A^{-1}_{[2]} \\ \vdots \\ A^{-1}_{[n]} \end{bmatrix}_{(n)\times(n+1)},$$

where $A^{-1}_{[k]}$ represents the k^{th} row of A^{-1}, for $k = 1, 2, \dots, n$.

2.3 Generalized Bernstein operational matrix of derivatives

The generalized n^{th} order of BOMD is given by [18]:

$$\frac{d^n}{dx^n} B(x) = \left(D^{(1)}\right)^n B(x), \qquad n = 1, 2, \dots . \tag{9}$$

Order one did not yield accurate results due to a single term present in the equation; hence, it was not applied in the study. For simplicity, the step for order one is omitted and only the generalizations for orders two and three are shown, as these are the orders applied in this study.

For $n = 2$, by using Equation (6) yields

$$B(x) = \left[b_0^2(x), b_1^2(x), b_2^2(x)\right],$$

where

$$b_0^2(x) = \begin{pmatrix} 2 \\ 0 \end{pmatrix} x^0(1-x)^2 = (1-x)^2,$$
$$b_1^2(x) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} x^1(1-x)^1 = 2x(1-x),$$
$$b_2^2(x) = \begin{pmatrix} 2 \\ 2 \end{pmatrix} x^2(1-x)^0 = x^2.$$

Hence,

$$B(x) = \begin{pmatrix} (1-x)^2 \\ 2x(1-x) \\ x^2 \end{pmatrix}.$$

Next, to find the derivative $D^{(1)}$, the following matrices are derived from Equation (8):

$$A = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 1 \end{pmatrix}_{3\times3}, \qquad H = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 2 \end{pmatrix}_{3\times2},$$

$$A^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & \frac{1}{2} & 1 \\ 0 & 0 & 1 \end{pmatrix}_{3\times3}, \qquad \hat{B} = \begin{pmatrix} A^{-1}_{[1]} \\ A^{-1}_{[2]} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & \frac{1}{2} & 1 \end{pmatrix}_{2\times3}.$$

By using the $D^{(1)} = AH\hat{B}$, derivative matrix for order two is given by

$$\begin{pmatrix} 1 & -2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & \frac{1}{2} & 1 \end{pmatrix} = \begin{pmatrix} -2 & -1 & 0 \\ 2 & 0 & -2 \\ 0 & 1 & 2 \end{pmatrix}.$$

For $n = 3$, by using Equation (6) yields

$$B(x) = \left[b_0^3(x), b_1^3(x), b_2^3(x), b_3^3(x)\right],$$

where

$$\begin{aligned} b_0^3(x) &= \begin{pmatrix} 3 \\ 0 \end{pmatrix} x^0(1-x)^3 = (1-x)^3, \\ b_1^3(x) &= \begin{pmatrix} 3 \\ 1 \end{pmatrix} x^1(1-x)^2 = 3x(1-x)^2, \\ b_2^3(x) &= \begin{pmatrix} 3 \\ 2 \end{pmatrix} x^2(1-x)^1 = 3x^2(1-x), \\ b_3^3(x) &= \begin{pmatrix} 3 \\ 3 \end{pmatrix} x^3(1-x)^0 = x^3. \end{aligned}$$

Hence,

$$B(x) = \begin{pmatrix} (1-x)^3 \\ 3x(1-x)^2 \\ 3x^2(1-x) \\ x^3 \end{pmatrix}.$$

Next, to find the derivative $D^{(1)}$, the following matrices are derived from Equation (8):

$$A = \begin{pmatrix} 1 & -3 & 3 & -1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix}_{4\times4}, \qquad H = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}_{4\times3},$$

$$A^{-1} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & \frac{1}{3} & \frac{2}{3} & 1 \\ 0 & 0 & \frac{1}{3} & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}_{4\times4}, \qquad \hat{B} = \begin{pmatrix} A^{-1}_{[1]} \\ A^{-1}_{[2]} \\ A^{-1}_{[3]} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & \frac{1}{3} & \frac{2}{3} & 1 \\ 0 & 0 & \frac{1}{3} & 1 \end{pmatrix}_{3\times4}.$$

By using the $D^{(1)} = AH\hat{B}$, derivative matrix for order three is given by

$$\begin{pmatrix} 1 & -3 & 3 & -1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & \frac{1}{3} & \frac{2}{3} & 1 \\ 0 & 0 & \frac{1}{3} & 1 \end{pmatrix} = \begin{pmatrix} -3 & -1 & 0 & 0 \\ 3 & -1 & -2 & 0 \\ 0 & 2 & 1 & -3 \\ 0 & 0 & 1 & 3 \end{pmatrix}.$$

From Equation (3) and Equation (9), the following equations are produced:

$$y(x) = \sum_{i=0}^{n} c_i b_i^n(x) = C^T B(x).$$

Thus,

$$\begin{aligned} Dy(x) &= C^T D^1 B(x), \\ D^2 y(x) &= C^T D^2 B(x). \end{aligned} \tag{10}$$

2.4 Collocation point

To get the solution of $y(x)$, the equation problem is collocated at n suitable points. The collocation points used in this study are the roots of Chebyshev polynomials. After testing with various collocating nodes, it is observed that this collocating technique yielded the best result. The roots of Chebyshev polynomials are given by [3]:

$$x_i = \frac{1}{2} + \frac{1}{2}\cos\left(\frac{(2i+1)\pi}{2n}\right), \quad i = 0, 1, \ldots, n-1. \tag{11}$$

A set of algebraic equations are produced from Equation (10) and Equation (11). By solving the unknown coefficient vector $C^T = [c_0, c_1, c_2, \ldots, c_n]$, the numerical solution of function $y(x)$ is obtained.

3 Numerical Results and Discussion

This section compares the proposed method to the exact solution and other existing numerical methods to test its accuracy and efficiency. Four numerical examples and a real-world problem are used to evaluate the method. The methods developed in Section 2 for the Bernstein polynomial of order $n = 2$ and $n = 3$ will be applied on the problems. The algebraic manipulations are done in Wolfram Mathematica Version 13.0.1.0 running on AMD Ryzen 5 6600H CPU 3.30GHz processor, 8GB RAM, with the digits rounded off to five decimal places for all tables of numerical values.

The Weierstrass approximation theorem [17] together with the Bernstein polynomials [22] are used to show the convergence of the proposed method, which leads the desired choice of parameter.

Theorem 3.1. *Suppose $y(x)$ is a continuous function on the interval $[0,1]$ and*

$$B_n(y,x) = \sum_{i=0}^{n} B_n^i(x) y\left(\frac{i}{n}\right), \tag{12}$$

is the Bernstein polynomial of degree n in terms of Bernstein basis, then $B_n(y,x)$ converges uniformly to $y(x)$.

The proof is omitted in this work. Readers may go through the work [22] for deeper understanding.

3.1 Problem 1: Application to linear equation

Consider the following linear singular two-point BVP as in [13]:

$$y'' + \frac{1}{x}y' + y = \frac{5}{4} + \frac{x^2}{16}, \tag{13}$$

subject to the boundary conditions

$$y'(0) = 0, \quad y(1) = \frac{17}{16}. \tag{14}$$

Equation (13) has the exact solution

$$y(x) = 1 + x^2/16.$$

Applying the method described in Section 2 for $n = 2$ yields

$$y(x) = c_0 b_0^2(x) + c_1 b_1^2(x) + c_2 b_2^2(x) = C^T B(x).$$

By using $y(x) = C^T B(x)$, $y'(x) = C^T DB(x)$ and $y''(x) = C^T D^2 B(x)$, the following equation is obtained:

$$C^T D^2 B(x) + \frac{1}{x} C^T DB(x) + C^T B(x) = \frac{5}{4} + \frac{x^2}{16}. \tag{15}$$

By collocating Equation (15) at the 2 collocation points for $n = 2$ given by

$$x_1 = \frac{1}{2} + \frac{1}{2\sqrt{2}}, \qquad x_2 = \frac{1}{2} - \frac{1}{2\sqrt{2}},$$

obtained by using Equation (11) resulted in

$$\begin{cases} \dfrac{(2+31\sqrt{2})c_0 - 2(30+31\sqrt{2})c_1 + (74+39\sqrt{2})c_2}{8(2+\sqrt{2})} = 1.2955, \\[2ex] \dfrac{(-2+31\sqrt{2})c_0 + (60-62\sqrt{2})c_1 + (-74+39\sqrt{2})c_2}{8(-2+\sqrt{2})} = 1.2513. \end{cases} \tag{16}$$

Applying the boundary conditions from Equation (14) yields

$$y'(0) = C^T DB(0) = 0, \qquad \text{which implies} \qquad c_1 = c_0,$$

$$y(1) = C^T B(1) = \frac{17}{16}, \qquad \text{which implies} \qquad c_2 = \frac{17}{16} = 1.0625.$$

Substituting the values c_0 and c_2 in Equation (16), $c_1 = c_0 = 1$ are obtained. Therefore, the coefficient values are

$$c_0 = 1, \qquad c_1 = 1, \qquad c_2 = \frac{17}{16}.$$

Hence, the equation obtained is given by

$$y(x) = \begin{pmatrix} 1 & 1 & \frac{17}{16} \end{pmatrix} \begin{pmatrix} (1-x)^2 \\ 2x(1-x) \\ x^2 \end{pmatrix} = 1 + \frac{x^2}{16},$$

which is the exact solution.

Table 1 lists the solution by applying the present method of order two, the exact solution and its absolute errors for different values of x ranging from $[0, 1]$. It is evident from Table 1 that the suggested approach delivers remarkably precise and reliable results that match the exact solution. It is also observed that the BOMD method produced more accurate results when compared to the results of He's VIM method for $n = 2$, as demonstrated in [13].

Figure 1 shows the curves of exact solutions, BOMD solutions for $n = 2$ and He's VIM method for $n = 2$. An observation of Figure 1 leads to the conclusion that the results obtained through the present method are highly consistent with the exact solution. The lines appeared to be overlapping, indicating a very good approximation to the exact solution. Figure 2 also shows the absolute error comparison between the proposed BOMD method and the comparison to He's VIM method. Since there is no error for the BOMD method, the graphical line does not appear in this example when plotted in loglog format. Therefore, we conclude that the method is applicable and efficient for solving linear equations.

Table 1: Numerical results by using BOMD $n = 2$ and comparison with He's VIM $n = 2$ for Problem 1.

x	Exact	BOMD $n = 2$		He's VIM $n = 2$ [13]	
	$y(x)$	$y(x)$	Error	$y(x)$	Error
0.1	1.00063	1.00063	0.	1.00077	1.41369×10^{-04}
0.2	1.00250	1.00250	0.	1.00264	1.40303×10^{-04}
0.3	1.00563	1.00563	0.	1.00576	1.38473×10^{-04}
0.4	1.01000	1.01000	0.	1.01014	1.35667×10^{-04}
0.5	1.01563	1.01563	0.	1.01576	1.31309×10^{-04}
0.6	1.02250	1.02250	0.	1.02262	1.24193×10^{-04}
0.7	1.03063	1.03063	0.	1.03074	1.12128×10^{-04}
0.8	1.04000	1.04000	0.	1.04009	9.15102×10^{-04}
0.9	1.05063	1.05063	0.	1.05068	5.68122×10^{-04}
1.0	1.06250	1.06250	0.	1.06250	3.10862×10^{-15}

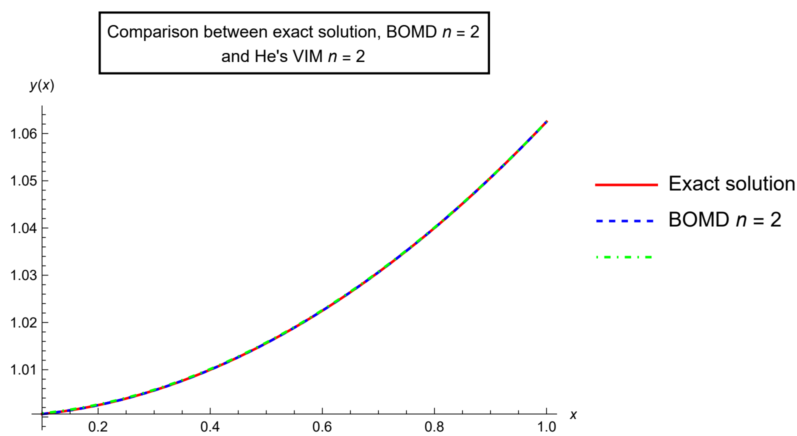

Figure 1: Graph comparison between exact solution, BOMD $n = 2$ and He's VIM $n = 2$ for Problem 1.

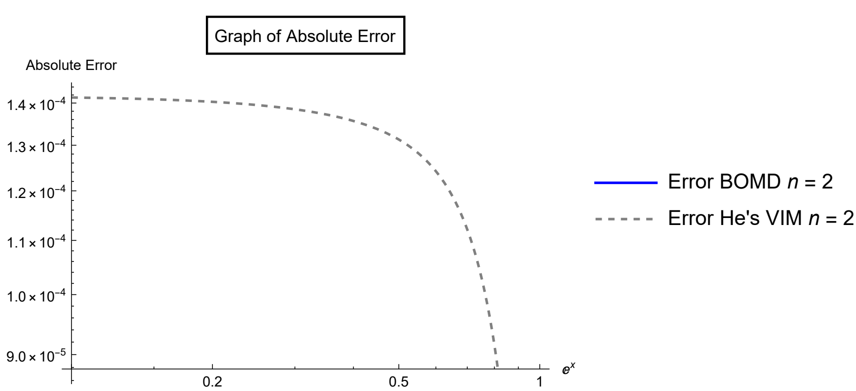

Figure 2: Graph of absolute error between BOMD $n = 2$ and He's VIM $n = 2$ for Problem 1.

3.2 Problem 2: Application to non-linear equation

Consider the following non-linear singular two-point BVP described by [22]:

$$-\left(x^2y'\right)' = x^2\left(-1+\frac{324}{53}x+\frac{54}{53}x^3-\frac{729}{2809}x^6+y^2\right), \qquad 0<x<1, \tag{17}$$

subject to the boundary conditions

$$y'(0)=0, \quad y(1)=\frac{1}{2}y\left(\frac{1}{3}\right). \tag{18}$$

Equation (17) has the exact solution $y(x) = -27/53x^3 + 1$.

The order for this problems starts with $n = 3$, as our algorithm did not achieve a solution when $n = 2$. An extended study should be made to solve and overcome this limitation. The steps for $n = 2$ are omitted for simplicity and we proceed to the following order.

Applying the method described in Section 2 for $n = 3$ yields

$$y(x) = c_0b_0^3(x) + c_1b_1^3(x) + c_2b_2^3(x) = C^TB(x).$$

By using $y(x) = C^TB(x)$, $y'(x) = C^TDB(x)$ and $y''(x) = C^TD^2B(x)$ and collocating Equation (17) at the 3 collocation points for $n = 3$ given by

$$x_0=\frac{1}{2}+\frac{\sqrt{3}}{4}, \quad x_1=\frac{1}{2}, \quad x_2=\frac{1}{2}-\frac{\sqrt{3}}{4},$$

obtained by using Equation (11), resulted in

$$\begin{cases} \dfrac{1}{65536}\left(\begin{array}{l} (-97+56\sqrt{3})c_0^2-9c_1^2-9(97+56\sqrt{3})c_2^2 \\ +2c_0\left(\begin{array}{l} -6144\sqrt{3}+3(-7+4\sqrt{3})c_1-3c_2 \\ -(7+4\sqrt{3})c_3 \end{array}\right) \\ -6c_1\left(\begin{array}{l} 10240(2+\sqrt{3})+3(7+4\sqrt{3})c_2 \\ +(97+56\sqrt{3})c_3 \end{array}\right) \\ +6c_2\left(6144(12+7\sqrt{3})-(1351+780\sqrt{3})c_3\right) \\ +c_3\left(-12288(26+15\sqrt{3})-(18817+10864\sqrt{3})c_3\right) \end{array}\right) = 4.666, \\ \dfrac{3}{2}(c_1-c_3)-\dfrac{1}{256}(c_0+3(c_1+c_2)+c_3)^2 = 0.54498, \\ \dfrac{1}{65536}\left(\begin{array}{l} -((97+56\sqrt{3})c_0^2)-9c_1^2+9(-97+56\sqrt{3})c_2^2 \\ +6(-1351+780\sqrt{3})c_2c_3+(-18817+10864\sqrt{3})c_3^2 \\ -2c_0\left(\begin{array}{l} -6144\sqrt{3}+3(7+4\sqrt{3})c_1+3c_2 \\ +(7-4\sqrt{3})c_3 \end{array}\right) \\ +12288\left(\begin{array}{l} 5(-2+\sqrt{3})c_1+(36-21\sqrt{3})c_2 \\ +(-26+15\sqrt{3})c_3 \end{array}\right) \\ +6c_1\left(3(-7+4\sqrt{3})c_2\right)+\left(-97+56\sqrt{3})c_3\right) \end{array}\right) = -0.00265. \end{cases} \tag{19}$$

Applying the boundary conditions from Equation (18) yields

$$y'(0) = C^TDB(0) = 0, \qquad \text{which implies} \qquad c_1 = c_0,$$
$$y(1) = C^TB(1) = \frac{1}{2}y\left(\frac{1}{3}\right), \qquad \text{which implies} \qquad c_3 = \frac{8c_0}{53}+\frac{12c_1}{53}+\frac{6c_2}{53}.$$

Substituting the values c_1, c_3 in Equation (19) and solving the equations resulted in

$$c_0 = 1, \quad c_2 = 1.$$

Therefore, the coefficient values are

$$c_0 = 1, \quad c_1 = 1, \quad c_2 = 1 \quad c_3 = \frac{26}{53}.$$

Hence, the equation obtained is given by

$$y(x) = \begin{pmatrix} 1 & 1 & 1 & \frac{26}{53} \end{pmatrix} \begin{pmatrix} (1-x)^3 \\ 3x(1-x)^2 \\ 3x^2(1-x) \\ x^3 \end{pmatrix} = 1 - \frac{27}{53}x^3,$$

which is the exact solution.

The approximate solution using the presented method of BOMD with $n = 3$ is compared with the exact solution and He's VIM $n = 3$ in [22]. The comparison is shown graphically in Figure 3 and Table 2 provides a comprehensive presentation of the numerical outputs of the solutions at varying values of x. Table 2 shows that the proposed method yields the exact result, as the absolute error values are zeros. Therefore, it is concluded that the BOMD method for $n = 3$ has better accuracy than He's VIM method for this problem.

Figure 3 shows the curves of exact solutions and the BOMD solutions for $n = 3$. The proposed method yields results that are identical to the exact solution and the two lines appear to be overlapping. Additionally, Figure 4 depicts the absolute error plot of the proposed approach. Similar to previous problem 1, the graphical line does not appear since the error is zero.

No graphical representation for He's VIM method for $n = 3$ is shown due to the absence of an equation solution for the method provided in the literature. However, from the numerical values in Table 2 and Figures 3 and 4 below, it can be observed that the absolute error value for the BOMD method is lower than He's VIM method. Hence, the proposed method is practical and efficient for solving non-linear equations.

Table 2: Numerical results by using BOMD $n = 3$ and comparison with He's VIM $n = 3$ for Problem 2.

x	Error	BOMD $n = 3$		He's VIM $n = 3$ [22]	
	$y(x)$	$y(x)$	Error	$y(x)$	Error
0.1	0.99949	0.99949	0.	0.99956	6.4×10^{-5}
0.2	0.99592	0.99592	0.	0.99599	6.4×10^{-5}
0.3	0.98625	0.98625	0.	0.98631	6.2×10^{-5}
0.4	0.96740	0.96740	0.	0.96746	6.1×10^{-5}
0.5	0.93632	0.93632	0.	0.93638	5.9×10^{-5}
0.6	0.88996	0.88996	0.	0.89002	5.7×10^{-5}
0.7	0.82526	0.82526	0.	0.82532	5.4×10^{-5}
0.8	0.73917	0.73917	0.	0.73922	4.9×10^{-5}
0.9	0.62862	0.62862	0.	0.62867	4.2×10^{-5}
1.0	0.49057	0.49057	0.	0.49060	3.1×10^{-5}

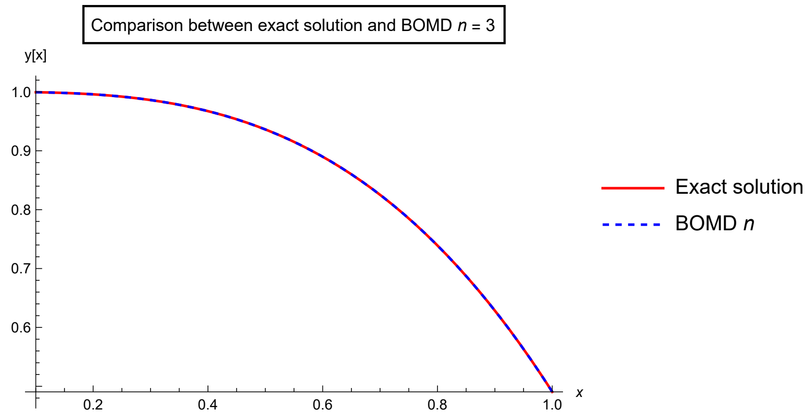

Figure 3: Graph comparison between exact solution and BOMD $n = 3$ for Problem 2.

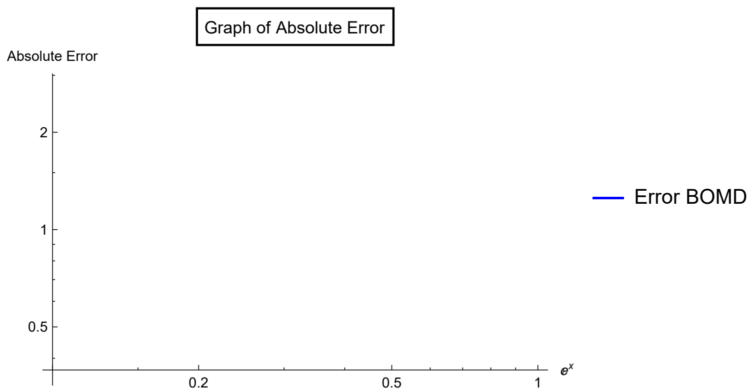

Figure 4: Graph of absolute error of BOMD $n = 3$ for Problem 2.

3.3 Problem 3: Application to linear system

Consider the following system of linear Ordinary Differential Equations (ODE) given by [3]:

$$\begin{aligned} y_1'(x) &= y_1(x) + y_2(x), \\ y_2'(x) &= -y_1(x) + y_2(x), \end{aligned} \tag{20}$$

subject to the initial conditions

$$y_1(0) = 0, \quad y_2(0) = 1, \tag{21}$$

with exact solutions $y_1(x) = e^x \sin(x), y_2(x) = e^x \cos(x)$.

By using the same step described in Section 2 for $n = 2$ and applying the boundary conditions from Equation (21) yields

$$\begin{aligned} y_1(0) &= C_1^T B(0) = 0, \qquad \text{which implies} \qquad c_{0,1} = 0, \\ y_2(0) &= C_2^T B(0) = 1, \qquad \text{which implies} \qquad c_{0,2} = 1. \end{aligned}$$

Solving all the coefficients resulted in

$$c_{0,1} = 0, \qquad c_{0,2} = 1, \qquad c_{1,1} = \frac{12}{25}, \qquad c_{1,2} = \frac{41}{25}, \qquad c_{2,1} = \frac{56}{25}, \qquad c_{2,2} = \frac{33}{25}.$$

Hence, the equation solution is given by

$$\begin{pmatrix} y_{1,2}(x) \\ y_{2,2}(x) \end{pmatrix} = \begin{pmatrix} c_{0,1} & c_{1,1} & c_{2,1} \\ c_{0,2} & c_{1,2} & c_{2,2} \end{pmatrix} \begin{pmatrix} (1-x)^2 \\ 2x(1-x) \\ x^2 \end{pmatrix} = \begin{pmatrix} \frac{24x}{25} + \frac{32x^2}{25} \\ 1 + \frac{32x}{25} - \frac{24x^2}{25} \end{pmatrix}.$$

Repeating the method for $n = 3$, the solution is given by

$$\begin{pmatrix} y_{1,2}(x) \\ y_{2,2}(x) \end{pmatrix} = \begin{pmatrix} c_{0,1} & c_{1,1} & c_{2,1} & c_{3,1} \\ c_{0,2} & c_{1,2} & c_{2,2} & c_{3,2} \end{pmatrix} \begin{pmatrix} (1-x)^3 \\ 3x(1-x)^2 \\ 3x^2(1-x) \\ x^3 \end{pmatrix} = \begin{pmatrix} \frac{1650x}{1681} + \frac{1920x^2}{1681} + \frac{288x^3}{1681} \\ 1 + \frac{1632x}{1681} + \frac{432x^2}{1681} - \frac{1280x^3}{1681} \end{pmatrix}.$$

Table 3 and Table 4 show that the error for the proposed method for $n = 2$ is higher than the Tau method [3]. To reduce the error for better accuracy, the order of the proposed method is increased to $n = 3$. As the order of the proposed method increases, its absolute error tends to decrease.

Figure 7 illustrates a comparison of exact solutions, BOMD solutions for $n = 3$ and the Tau method. At the same time, Figure 8 presents a comparison of the proposed method's absolute error for $n = 3$ and the Tau method for $y_1(x)$ and $y_2(x)$, respectively. It can be observed from Figure 5 that the line of BOMD for $y_2(x)$ does not overlap with and is diverging from the exact solution. However, by increasing the order to three, the line is now overlapping with the exact solution and no longer diverges, as shown in Figure 7.

Observe that the graph of absolute errors for the BOMD method is lower than the Tau method and shows better accuracy than the BOMD of order two. A hypothesis is made that increasing the proposed method's order to $n = 4$ might yield even more accurate results with minor errors. Therefore, the proposed method is applicable and efficient for solving linear systems.

Table 3: $y_1(x)$ comparison by using BOMD $n = 2$, BOMD $n = 3$ and Tau Method $n = 2$ for Problem 3.

x	Exact	BOMD $n=2$		BOMD $n=3$		Tau Method $n=2$ [3]	
	$y_1(x)$	$y_1(x)$	Error	$y_1(x)$	Error	$y_1(x)$	Error
0.1	0.11033	0.10880	1.53290×10^{-3}	0.10975	5.84030×10^{-4}	0.10615	4.17914×10^{-3}
0.2	0.24266	0.24320	5.44731×10^{-3}	0.24337	7.14154×10^{-4}	0.24000	2.65527×10^{-3}
0.3	0.39891	0.40320	4.28945×10^{-3}	0.40189	2.97880×10^{-3}	0.40154	2.62791×10^{-3}
0.4	0.58094	0.58880	7.85610×10^{-3}	0.58634	5.39280×10^{-3}	0.59077	9.82533×10^{-3}
0.5	0.79044	0.80000	9.56092×10^{-3}	0.79774	7.30036×10^{-3}	0.80769	1.72532×10^{-2}
0.6	1.02880	1.03680	7.95433×10^{-3}	1.03713	8.27985×10^{-3}	1.05231	2.34620×10^{-2}
0.7	1.29730	1.29920	1.90489×10^{-3}	1.30552	8.22779×10^{-3}	1.32462	2.73203×10^{-2}
0.8	1.59650	1.58720	9.30534×10^{-3}	1.60396	7.45421×10^{-3}	1.62462	2.81100×10^{-2}
0.9	1.92670	1.90080	2.58733×10^{-2}	1.93346	6.79011×10^{-3}	1.95231	2.56344×10^{-2}
1.0	2.28740	2.24000	4.73553×10^{-2}	2.29506	7.70718×10^{-3}	2.30769	2.03370×10^{-2}

Table 4: $y_2(x)$ comparison by using BOMD $n = 2$, BOMD $n = 3$ and Tau Method $n = 2$ for Problem 3.

x	Exact	BOMD $n=2$		BOMD $n=3$		Tau Method $n=2$ [3]	
	$y_2(x)$	$y_2(x)$	Error	$y_2(x)$	Error	$y_2(x)$	Error
0.1	1.09965	1.11840	1.87503×10^{-2}	1.09889	7.56151×10^{-4}	1.12923	2.95811×10^{-2}
0.2	1.19706	1.21760	2.05440×10^{-2}	1.19836	1.30210×10^{-3}	1.24000	4.29440×10^{-2}
0.3	1.28957	1.29760	8.03063×10^{-3}	1.29383	4.25573×10^{-3}	1.33231	4.27383×10^{-2}
0.4	1.37406	1.35840	1.56615×10^{-2}	1.38073	6.66422×10^{-3}	1.40615	3.20923×10^{-2}
0.5	1.44689	1.40000	4.68890×10^{-2}	1.45449	7.60234×10^{-3}	1.46154	1.46494×10^{-2}
0.6	1.50386	1.42240	8.14595×10^{-2}	1.51055	6.69370×10^{-3}	1.49846	5.39800×10^{-2}
0.7	1.54020	1.42560	1.14603×10^{-1}	1.54434	4.13963×10^{-3}	1.51692	2.32799×10^{-2}
0.8	1.55055	1.40960	1.40949×10^{-1}	1.55129	7.41601×10^{-4}	1.51692	3.36262×10^{-2}
0.9	1.52891	1.37440	1.54514×10^{-1}	1.52683	2.08454×10^{-3}	1.49846	3.04523×10^{-2}
1.0	1.46869	1.32000	1.48694×10^{-1}	1.46639	2.30489×10^{-3}	1.46154	7.15548×10^{-3}

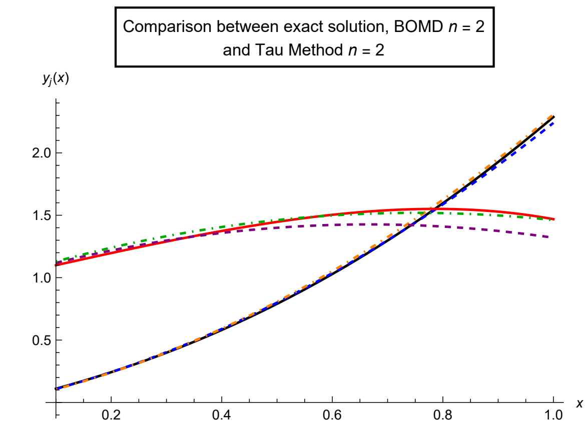

Figure 5: Graph comparison between exact solution, BOMD $n = 2$ and Tau method $n = 2$ for Problem 3.

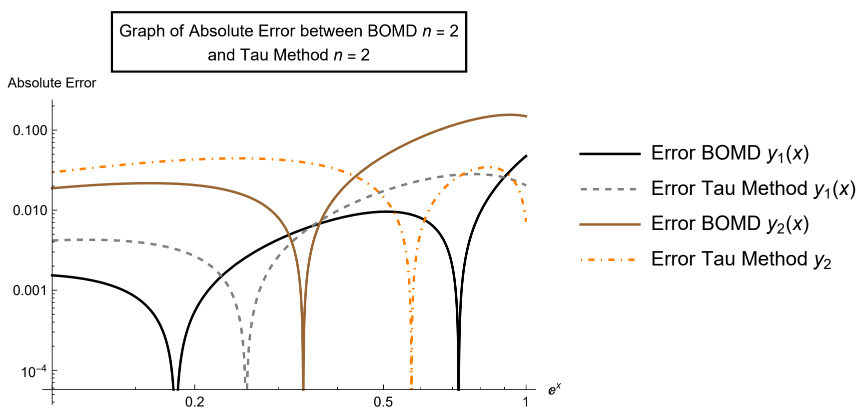

Figure 6: Graph of absolute error of BOMD $n = 2$ and Tau Method $n = 2$ for Problem 3.

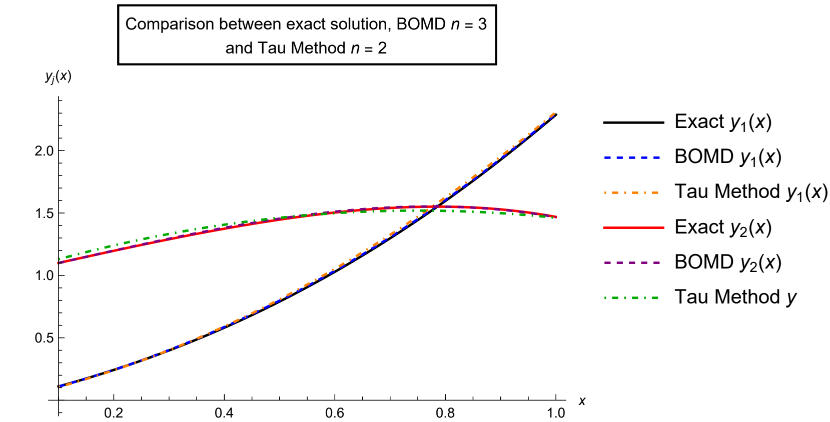

Figure 7: Graph comparison between exact solution, BOMD $n = 3$ and Tau method $n = 2$ for Problem 3.

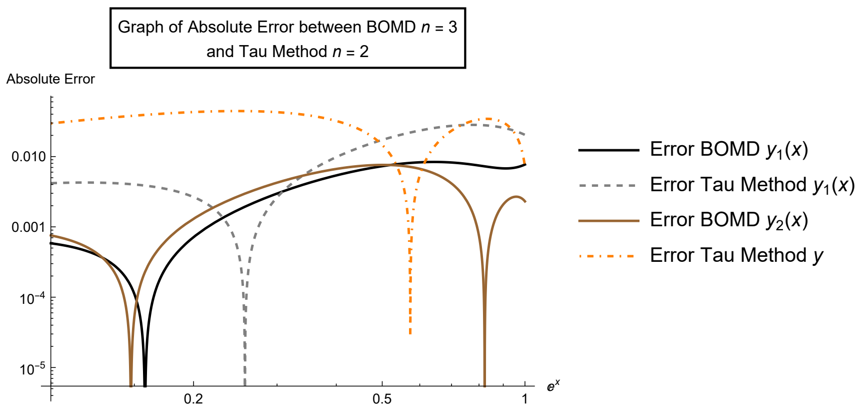

Figure 8: Graph of absolute error of BOMD $n = 3$ and Tau method $n = 2$ for Problem 3.

3.4 Problem 4: Application to non-linear system

Consider the following stiff system of non-linear ODE given by [3]:

$$\begin{aligned} y_1'(x) &= -1002y_1(x) + 1000y_2^2(x), \\ y_2'(x) &= y_1(x) - y_2(x) - y_2^2(x), \end{aligned} \tag{22}$$

subject to the initial conditions

$$y_1(0) = 1, \quad y_2(0) = 1, \tag{23}$$

and the exact solutions $y_1(x) = e^{-2x}, y_2(x) = e^{-x}$.

By using the same step described in the previous section for $n = 2$ and applying the initial conditions from Equation (23) yields

$$y_1(0) = C_1^T B(0) = 1, \qquad \text{which implies} \qquad c_{0,1} = 1,$$

$$y_2(0) = C_2^T B(0) = 1, \qquad \text{which implies} \qquad c_{0,2} = 1.$$

Solving the equations, the values of the coefficients yields are

$$c_{0,1}=1,\qquad c_{0,2}=1,\qquad c_{1,1}=0.072316,$$
$$c_{1,2}=0.520046,\qquad c_{2,1}=0.180287,\qquad c_{2,2}=0.359918.$$

Hence, the equation solution is given by

$$\begin{pmatrix} y_{1,2}(x) \\ y_{2,2}(x) \end{pmatrix} = \begin{pmatrix} c_{0,1} & c_{1,1} & c_{2,1} \\ c_{0,2} & c_{1,2} & c_{2,2} \end{pmatrix} \begin{pmatrix} (1-x)^2 \\ 2x(1-x) \\ x^2 \end{pmatrix} \approx \begin{pmatrix} 1-1.855368x+1.035656x^2 \\ 1-0.959908x+0.319825x^2 \end{pmatrix}.$$

Repeating the method for $n=3$, the equation solution is given by

$$\begin{pmatrix} y_{1,2}(x) \\ y_{2,2}(x) \end{pmatrix} = \begin{pmatrix} c_{0,1} & c_{1,1} & c_{2,1} & c_{3,1} \\ c_{0,2} & c_{1,2} & c_{2,2} & c_{3,2} \end{pmatrix} \begin{pmatrix} (1-x)^3 \\ 3x(1-x)^2 \\ 3x^2(1-x) \\ x^3 \end{pmatrix}$$
$$\approx \begin{pmatrix} 1-1.983931x+1.759629x^2-0.647224x^3 \\ 1-0.996722x+0.468935x^2-0.104127x^3 \end{pmatrix}.$$

Table 5 and Table 6 exhibit a comparison between the absolute errors of the present method and the Tau method. The error for the proposed method for $n=2$ is observed to be higher than the results of the Tau method. Figure 9 illustrates the comparison of the curves representing the analytical solutions and those generated through the BOMD method for $n=2$. The order is increased to $n=3$ to improve the accuracy.

From Table 5 and Table 6, the proposed method for $n=3$ produces excellent results with smaller errors compared to the previous order. It can be observed from Figure 9 that the line of BOMD for $y_2(x)$ does not overlap and is diverging from the exact solution. Similar to the previous example, by increasing the order to three, the line of BOMD now overlaps with the exact solution and no longer diverges, as shown in Figure 11. Therefore, our previous hypothesis on the order of the proposed method is directly proportional to the accuracy of the solution remains true in this problem.

Table 5: $y_1(x)$ comparison by using BOMD $n=2$, BOMD $n=3$ and Tau Method $n=2$ for Problem 4.

x	Exact	BOMD $n=2$		BOMD $n=3$		Tau Method $n=2$ [3]	
	$y_1(x)$	$y_1(x)$	Error	$y_1(x)$	Error	$y_1(x)$	Error
0.1	0.81873	0.82482	6.08901×10^{-3}	0.81856	1.74787×10^{-4}	0.83430	1.55692×10^{-2}
0.2	0.67032	0.67035	3.25940×10^{-5}	0.66842	1.89888×10^{-3}	0.68720	1.68800×10^{-2}
0.3	0.54881	0.53660	1.22130×10^{-2}	0.54571	3.09937×10^{-3}	0.55870	9.88836×10^{-3}
0.4	0.44933	0.42356	2.57712×10^{-2}	0.44655	2.78306×10^{-3}	0.44880	5.28964×10^{-4}
0.5	0.36788	0.33123	3.66494×10^{-2}	0.36704	8.40691×10^{-4}	0.35750	1.03794×10^{-2}
0.6	0.30119	0.25962	4.15789×10^{-2}	0.30331	2.11324×10^{-3}	0.28480	1.63942×10^{-2}
0.7	0.24660	0.20871	3.78831×10^{-2}	0.25147	4.87171×10^{-3}	0.23070	1.58970×10^{-2}
0.8	0.20190	0.17853	2.33711×10^{-2}	0.20764	5.74255×10^{-3}	0.19520	6.69652×10^{-3}
0.9	0.16530	0.16905	3.75127×10^{-3}	0.16794	2.63641×10^{-3}	0.17830	1.30011×10^{-2}
1.0	0.13534	0.18029	4.49527×10^{-2}	0.12847	6.86128×10^{-3}	0.18000	4.46647×10^{-2}

Table 6: $y_2(x)$ comparison by using BOMD $n = 2$, BOMD $n = 3$ and Tau Method $n = 2$ for Problem 4.

x	Exact	BOMD $n=2$		BOMD $n=3$		Tau Method $n=2$ [3]	
	$y_2(x)$	$y_2(x)$	Error	$y_2(x)$	Error	$y_2(x)$	Error
0.1	0.90484	0.90721	2.37003×10^{-3}	0.90491	7.56050×10^{-5}	0.90810	3.26258×10^{-3}
0.2	0.81873	0.82081	2.08065×10^{-3}	0.81858	1.50769×10^{-4}	0.82240	3.66925×10^{-3}
0.3	0.74082	0.74081	6.37068×10^{-6}	0.74038	4.42100×10^{-4}	0.74290	2.08178×10^{-3}
0.4	0.67032	0.66721	3.11125×10^{-3}	0.66968	6.43374×10^{-4}	0.66960	7.20046×10^{-4}
0.5	0.60653	0.60000	6.52841×10^{-3}	0.60586	6.73785×10^{-4}	0.60250	4.03066×10^{-3}
0.6	0.54881	0.53919	9.61944×10^{-3}	0.54829	5.19668×10^{-4}	0.54160	7.21164×10^{-3}
0.7	0.49659	0.48478	1.18067×10^{-2}	0.49636	2.28115×10^{-4}	0.48690	9.68530×10^{-3}
0.8	0.44933	0.43676	1.25674×10^{-2}	0.44943	9.88119×10^{-5}	0.43840	1.09290×10^{-2}
0.9	0.40657	0.39514	1.14286×10^{-2}	0.40688	3.09307×10^{-4}	0.39610	1.04697×10^{-2}
1.0	0.36788	0.35992	7.96244×10^{-3}	0.36809	2.06559×10^{-4}	0.36000	7.87944×10^{-3}

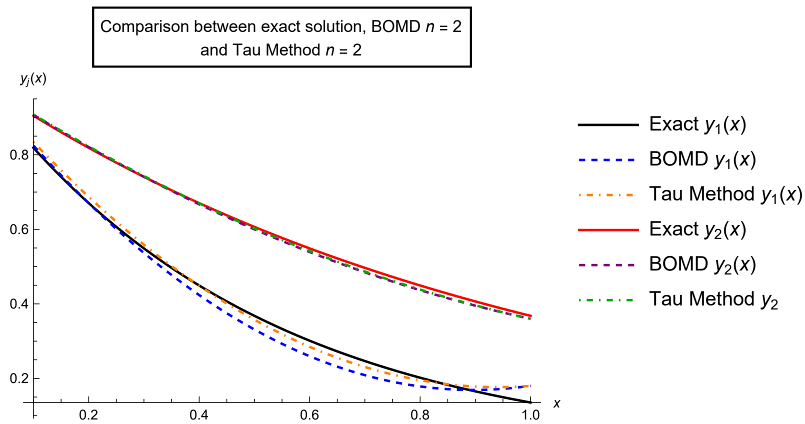

Figure 9: Graph comparison between exact solution, BOMD $n = 2$ and Tau method $n = 2$ for Problem 4.

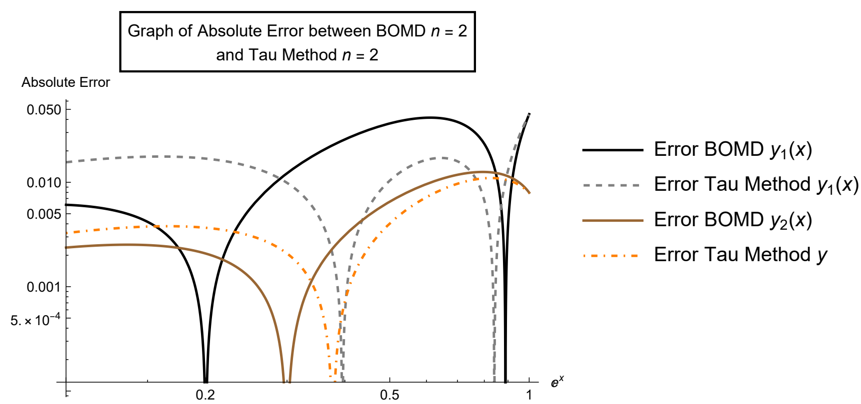

Figure 10: Graph of absolute error of BOMD $n = 2$ and Tau Method $n = 2$ for Problem 4.

The plot of absolute errors of the proposed method for $n = 3$ and the comparison method is also shown in Figure 12. Based on the data presented in Figure 12, it can be observed that the outcome of the proposed method correspond to the analytical solution, thereby making it more relevant and effective when dealing with non-linear systems.

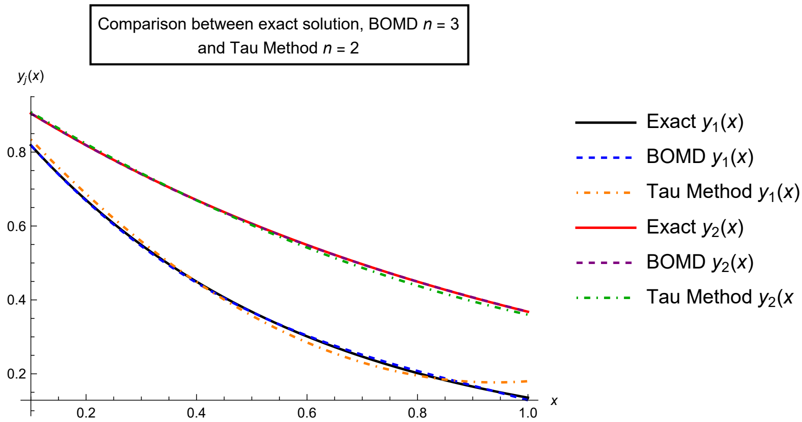

Figure 11: Graph comparison between exact solution, BOMD $n = 3$ and Tau method $n = 2$ for Problem 4.

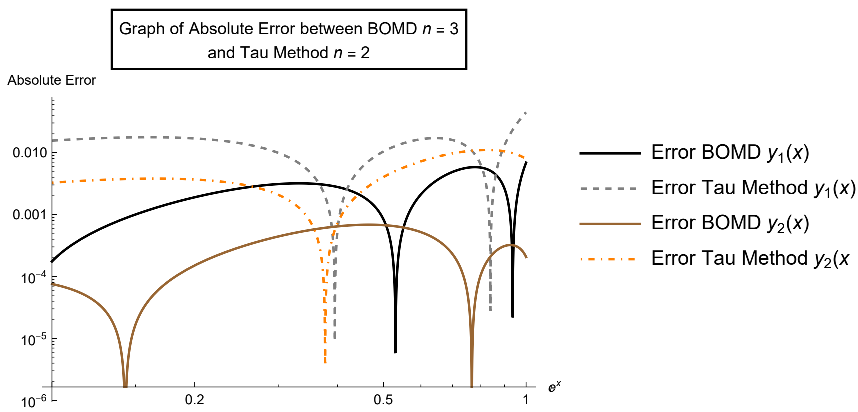

Figure 12: Graph of absolute error of BOMD $n = 3$ and Tau method $n = 2$ for Problem 4.

3.5 Problem 5: Application to real-world problem

Consider the following problem described by [5]:

$$P'(t) = 0.0009906(1000 - P(t))P(t), \tag{24}$$

subject to the condition

$$P(0) = 1. \tag{25}$$

Equation (24) has the solution given by $P(t) = 1000/1 + 999e^{\text{-0.9906t}}$.

By applying the method developed for $n = 2$ and applying the boundary conditions from Equation (25) yields

$$P(0) = C^T B(0) = 0, \qquad \text{which implies} \qquad c_0 = 1.$$

Solving all the coefficients, the equation solution is thus given by

$$P(t) = 1 + 0.883566t + 0.861068t^2.$$

Repeating the method for $n = 3$, the solution is given by

$$P(t) = 1 + 0.997885t + 0.418762t^2 + 0.270301t^3.$$

Table 7 shows the numerical results of the approximate solution at interval $[0, 1]$, along with the given solution and the absolute error of the BOMD method. Figure 13 compares the given and the approximate solutions yielded using the proposed method of order two. Figure 14 shows the absolute error of the solution. Next, the BOMD of order three is applied to measure the accuracy of the suggested approach as the order increases.

Table 7: Numerical results by using BOMD $n = 2$, BOMD $n = 3$ and comparison with given solution for Problem 5.

t	Given	BOMD $n = 2$		BOMD $n = 3$	
	$P(t)$	$P(t)$	Error	$P(t)$	Error
0.1	1.10402	1.09697	7.05030×10^{-3}	1.10425	2.28879×10^{-4}
0.2	1.21884	1.21116	7.68570×10^{-3}	1.21849	3.51654×10^{-4}
0.3	1.34559	1.34257	3.02600×10^{-3}	1.34435	1.23959×10^{-3}
0.4	1.48550	1.49120	5.69360×10^{-3}	1.48346	2.04834×10^{-3}
0.5	1.63994	1.65705	1.71107×10^{-2}	1.63742	2.51846×10^{-3}
0.6	1.81040	1.84012	2.97229×10^{-2}	1.80787	2.53060×10^{-3}
0.7	1.99855	2.04042	4.18734×10^{-2}	1.99643	2.11969×10^{-3}
0.8	2.20620	2.25794	5.17357×10^{-2}	2.20471	1.49054×10^{-3}
0.9	2.43538	2.49267	5.72961×10^{-2}	2.43434	1.03494×10^{-3}
1.0	2.68830	2.74463	5.63352×10^{-2}	2.68695	1.35045×10^{-3}

By comparing the graph of Figure 13 and Figure 15, it is observed that the lines tend to be closer to the exact solution (overlapping) when the order of the method is higher. Figure 16 shows the absolute error of the solution for order three. Figure 17 shows that the absolute error for BOMD of order three is much lower than for order two. Thus, our hypothesis of higher accuracy as order increases stands. Therefore, the proposed method is applicable in solving real-world problems.

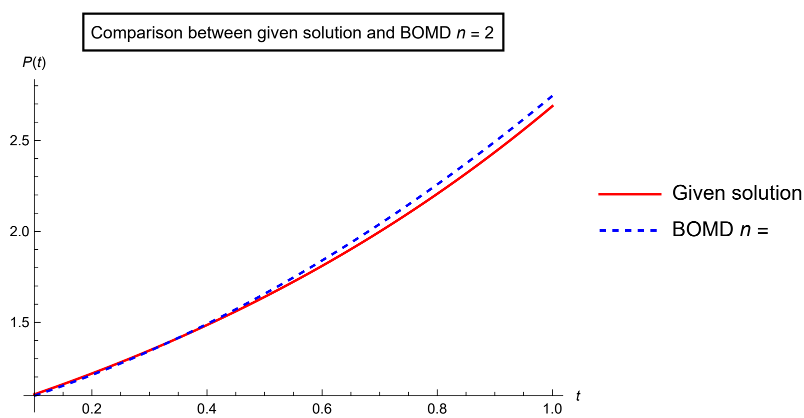

Figure 13: Graph comparison between given solution and BOMD $n = 2$ for Problem 5.

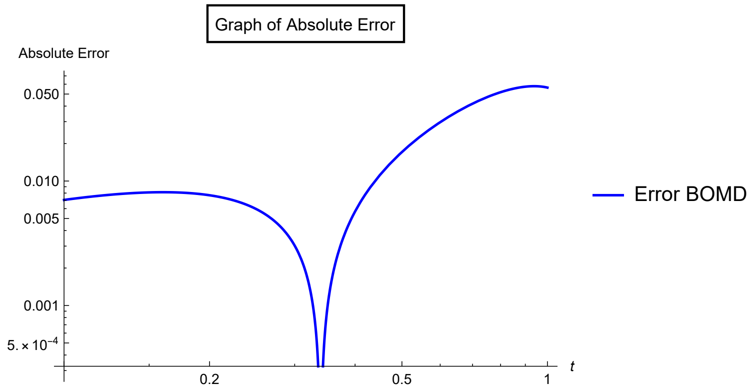

Figure 14: Graph of absolute error of BOMD $n = 2$ for Problem 5.

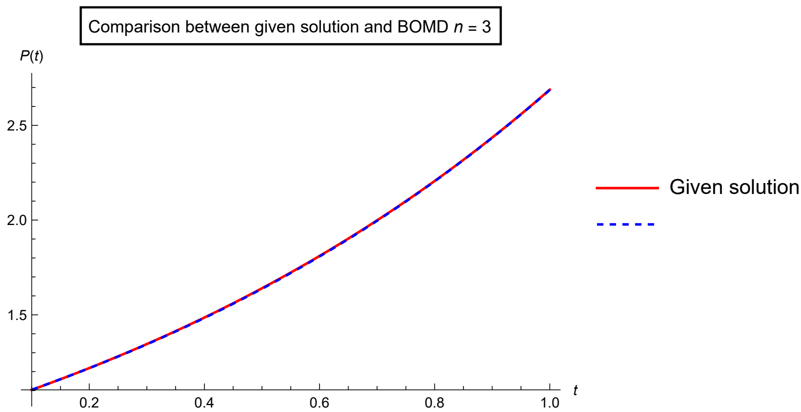

Figure 15: Graph comparison between given solution and BOMD $n = 3$ for Problem 5.

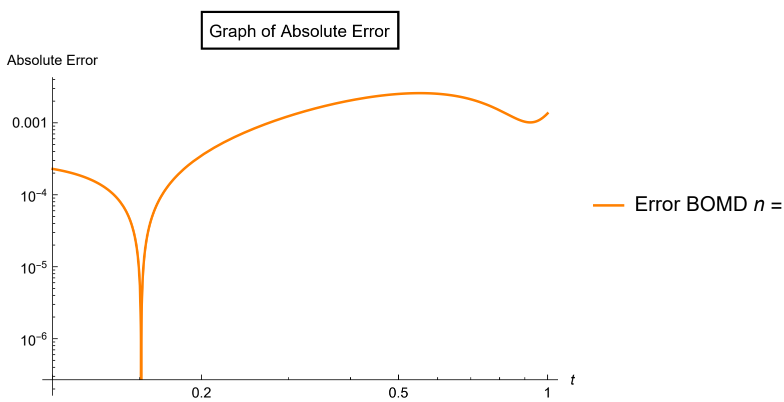

Figure 16: Graph of absolute error of BOMD $n = 3$ for Problem 5.

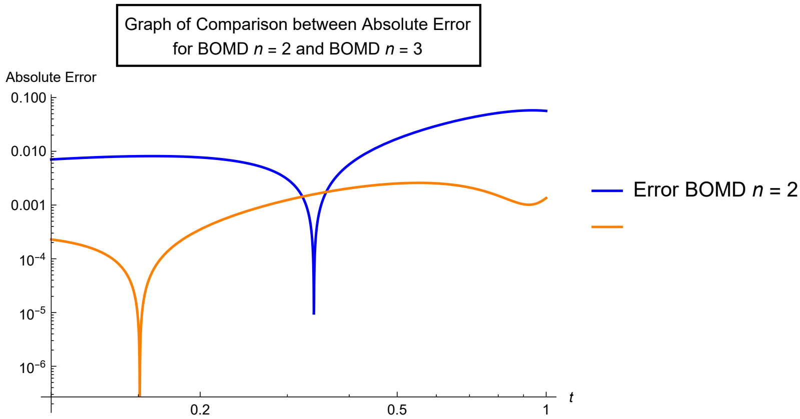

Figure 17: Graph of comparison between absolute error of BOMD $n = 2$ and BOMD $n = 3$ for Problem 5.

4 Conclusions

The accuracy of the proposed method is validated by examining and comparing four numerical examples and one real-world application with the exact solutions. An analysis of the proposed method as a solution to the four problems revealed that it is able to produce a more accurate and precise outcome than He's VIM and Tau methods by calculating the absolute error.

The results show that as the order of the method increased, the errors decreased; hence, the accuracy is improved. These results led us to conclude that the proposed method effectively solved both linear and non-linear equations and systems of two-point BVP, and is applicable in solving real-world problems. The computer program for the method is simple and easy to modify according to equation problems, making it more cost-effective. In conclusion, the BOMD method is a more efficient approach to implement in solving BVP and it yields a highly satisfactory result with only a small number of bases.

It is important to stress that the proposed method produced n equations as the number of collocation points used in solving the coefficient and finding the solution, which means the higher order of the proposed method requires higher cost and time in running the simulations. In some cases, for a particular value of n, the method does not manage to yield the coefficients. For future works, new approaches shall be explored to reduce computational work and overcome this shortcoming of the method, such as by deriving a new collocating point that suits the proposed method. By using the best collocating point, it is expected that the solution will converge to its exact solution in lesser time, therefore making it more efficient.

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Conflicts of Interest The authors declare no conflict of interest.

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